

Ascending Auctions and Walrasian Equilibrium

Oren Ben-Zwi*

Ron Lavi†

Ilan Newman‡

December 12, 2013

Abstract

We present a family of submodular valuation classes that generalizes gross substitute. We show that Walrasian equilibrium always exist for one class in this family, and there is a natural ascending auction which finds it. We prove some new structural properties on gross-substitute auctions which, in turn, show that the known ascending auctions for this class (Gul-Stacchetti and Ausbel) are, in fact, identical. We generalize these two auctions, and provide a simple proof that they terminate in a Walrasian equilibrium.

1 Introduction

Ascending auctions are an important auction format with many advantages such as visibility of price formation and gradual disclosure of valuations. It is harder for a dishonest auctioneer to manipulate the bids in such auctions, and it is easier to adjust such auctions to a setting with budget constraints. It is a popular auction format in practice, and its theoretical properties have been widely studied.

A significant part of the literature tries to characterize settings where the end outcome of the ascending auction is a Walrasian equilibrium: this is an outcome where each bidder is being allocated her most preferred subset of items, under final prices, and, furthermore, all items are being allocated. Such an outcome is desirable as it provides some stability, and additionally it guarantees optimal social welfare. Kelso and Crawford [16] define the class of “gross substitutes” (GS) valuations, a small subclass of submodular valuations that ensures the existence of a Walrasian equilibrium. They argue that a natural ascending auction terminates in a Walrasian equilibrium, when valuations are GS and bidders report true demands throughout the auction.

A complete treatment of ascending auctions for gross substitutes valuations is offered by Gul and Stacchetti [15]. They give a formal, specific ascending auction and prove that it terminates in a Walrasian equilibrium when valuations are GS. They additionally show in a companion paper [14] that GS is the unique maximal class of valuations that contains unit-demand valuations and guarantees the existence of a Walrasian equilibrium: for any non-GS valuation there exist unit-demand valuations such that a Walrasian equilibrium does not exist for these unit-demand valuations coupled with the given non-GS valuation.

In this paper we study whether it may be possible to construct an ascending auction that reaches a Walrasian equilibrium for submodular valuations that are not GS. This question is twofold: First, what are natural subclasses of submodular valuations for which a Walrasian equilibrium is guaranteed to exist? Second, for these valuation classes, can we reach a Walrasian equilibrium via an ascending auction? We should remark that given the existing literature on ascending auctions described above, this question may initially seem hopeless. After all, GS is the maximal class that ensures existence of Walrasian equilibrium. However, in this paper we demonstrate that this intuition is false. Our main result shows another natural subclass of submodular valuations that ensures the existence of a Walrasian equilibrium, and gives an ascending auction that finds this equilibrium outcome.

*University of Vienna Faculty of Computer Science Vienna, Austria oren.ben-zwi@univie.ac.at

†Technion – Israel Institute of Technology Faculty of Industrial Engineering and Management ronlavi@ie.technion.ac.il

‡University of Haifa Department of Computer Science ilan@cs.haifa.ac.il

Main Result: There exists a subclass of submodular valuations that is not GS and (still) guarantees the existence of a Walrasian equilibrium. Furthermore, there exists an ascending auction that always terminates in a Walrasian equilibrium, if all valuations belong to this class.

Recall that a unit-demand valuation assigns arbitrary values to singletons, and the value of every subset of items of cardinality at least two is equal to the largest value of a singleton in this subset. The new class that we define is a “unit demand with a twist”, as follows. Fix some arbitrary number M . Then in our valuation class, the value of every subset of items of cardinality at least two is equal to M (and we include all unit demand valuations to which this change does not violate monotonicity and sub-modularity). This is a very simple class and it is not GS. Nevertheless, a Walrasian equilibrium always exists for any tuple of valuations taken from this class, and it can be obtained via an ascending auction.

Our ascending auction is a careful generalization of the Gul-Stacchetti auction. Although their auction is widely cited, the proof that they give is long and hard to follow. This has been acknowledged several times in the past, for example by Ausubel [2] who proposes a different auction formulation, with a different analysis. Developing tools and intuition for the main analysis of this paper yields basic structural results for gross substitute valuations, that shed new light on both these auctions and their proofs. First, we show that both these auctions are completely equivalent (while their description is significantly different). More specifically, we show that in every step, both auctions raise the prices of exactly the same items. Second, we give a class of ascending auctions for gross substitutes valuations that contains the Gul-Stacchetti auction as well as other (truly) alternative formats, and that terminate in a Walrasian equilibrium. Our analysis is much simpler (and shorter) than that of Gul and Stacchetti. Since the Gul-Stacchetti auction is an important widely cited result, we feel that a new concise full proof is of independent theoretical interest.

This new family of ascending auctions can be viewed as a family of primal dual algorithms. Bikhchandani and Mamer [6] show that a sufficient and necessary condition for the existence of a Walrasian equilibrium is that there will be no integrality gap in a certain linear program, usually termed the “winner determination linear program”. Ausubel explicitly uses the dual of this program to construct his auction, and by iteratively improving the dual solution he finds Walrasian prices. By incorporating part of his method into the proof of Gul and Stacchetti, we are able to both simplify the proof as well as generalize the result, showing a family of ascending auctions for gross substitutes.

We view our results as the beginning of a search for a full characterization of conditions that ensure both the existence of Walrasian equilibria in subclasses of submodular valuations, and the existence of ascending auctions that find these equilibria outcomes. As our results in this paper demonstrate, this is a complicated technical task. Still, relying on all that we have described above, we believe it is an important task.

1.1 Additional Related Literature

[9] were the first to formally define an ascending auction that terminates in a Walrasian equilibrium. They analyzed the case of unit-demand valuations. [1] fully characterize the class of all ascending auctions that terminate in a Walrasian equilibrium, for unit-demand bidders. It has been later recognized by [15] that the DGS auction can be generalized to gross substitutes valuations, as detailed above.

The existence of a Walrasian equilibrium for non-GS valuation classes was barely studied. [6] describe few cases of super-additive valuation classes, and conclude that the exploration of this issue is an important subject for future research. [20] identify a class of valuations with complements that ensure the existence of a Walrasian equilibrium, and later provide an iterative auction (that increases and decreases prices) that finds a Walrasian equilibrium for their class of valuations [21].

Several papers study strategic properties of ascending auctions, showing that truthful demand reporting is a Nash equilibrium in various types of ascending auctions. [3] shows this for the case of identical items, and more recently [5] show this when items form a basis of a matroid or a polymatroid.

As mentioned above, it is usually easy to adjust ascending auctions to cases where bidders have budget constraints, as studied by [10] and more recently by [8, 13].

Finally, we should mention that a strand of the literature that studies ascending auction with nonlinear bundle prices (where there is price for every subset of items), as initiated by Parkes [19] and by Ausubel and

Milgrom [4].

2 Preliminaries and Structural Results

In the following we denote by \mathbb{R}_+ the set of non-negative reals.

A *combinatorial auction* is a setting where a finite set of items, Ω , is to be partitioned between a set of players. In what follows it is always assumed that $|\Omega| = m$, and that there are n players associated with valuations v_i , $i = 1, \dots, n$, where $v_i : 2^\Omega \mapsto \mathbb{R}_+$. Namely, for every set $S \subseteq \Omega$, $v_i(S)$ measures how much the i th player favors the set of items S . Valuation are assumed here to be monotone with respect to inclusion (aka *free disposal*), and that $v_i(\emptyset) = 0$ for every i . The auctioneer sets a price for each item. This is given by a function $p : \Omega \mapsto \mathbb{R}_+$. We refer to p also as a price-vector. Having a price on single items, the price naturally extends additively to subsets, namely, for $S \subseteq \Omega$, $p(S) = \sum_{j \in S} p(j)$.

Having a set of valuations $\{v_i\}_1^n$ and a price vector p , the *utility* of player i from a set $S \subseteq \Omega$, denoted by $u_{i,p}(S)$ is defined by $u_{i,p}(S) = v_i(S) - p(S)$. The *demand* of a player i and price vector p is a collection of subsets of items of maximum utility. Namely, $D_i(p) = \{S \mid \forall S' \subseteq \Omega, u_{i,p}(S) \geq u_{i,p}(S')\}$.

Occasionally we omit subscripts when there is no risk of confusion, e.g., we use $u_i(S)$ instead of $u_{i,p}(S)$ when p is fixed or well defined. We use u_i or $u_{i,p}$ to denote the utility of player i , which is its utility from a demand set. So $u_i = u_i(S)$ for some $S \in D_i(p)$ when the price vector is known to be p .

An *allocation* of the items is a map $A : \{1, \dots, n\} \mapsto 2^\Omega$, which we also denote by (S_1, \dots, S_n) . Namely, where $A(i) = S_i \subseteq \Omega$. The requirement is that the sets $\{S_i\}_1^n$ are pairwise disjoint. We think of an allocation as allocating each player i , the items of the set S_i . S_i may be empty, for some $i \in [n] = \{1, 2, \dots, n\}$.

Definition 2.1 (Envy Free). *Given a price vector p , an allocation is envy free w.r.t. p if every player is allocated a demand set. A price vector is envy free if there exists an envy free allocation for it.*

As the name suggests, an envy-free allocation w.r.t. a price vector p signifies an economical situation of 'equilibrium' in the sense that under the corresponding price p , each player is satisfied with its allocated set. Clearly, for every set of valuations, there is an envy-free price vector and an envy-free allocation for it - simply take p large enough so that the demand set of every player contains just the empty set. This fact, and the discussion below motivates the following definition of a Walrasian allocation.

A *Walrasian allocation* [23] is an envy free allocation for which every *unallocated item*, has price zero. In other words, the union of the allocated sets cover all items of positive price. A price vector for which there exists a Walrasian allocation is called a *Walrasian price vector*.

The existence of a Walrasian price vector is not guaranteed for every set of valuations. If it exists it is said to be Walrasian equilibrium and the set of valuation is said to poses a Walrasian equilibrium. There is another importance of the Walrasian equilibrium - if it exists, it is known to maximize the so called *social welfare* among all allocations. The social welfare is $\sum_{i \in [n]} v_i(S_i)$, the sum of the individual values of the sets that are allocated. The maximum social welfare can be written as a solution to the following *winner determination* integer linear program (ILP), $P1$:

$$\max \left\{ \sum_{i,S} v_i(S) \cdot x_{i,S} \right\} \tag{1}$$

$$s.t. \quad \forall j \in \Omega, \quad \sum_{i,S \mid j \in S} x_{i,S} \leq 1 \tag{2}$$

$$\forall i \in [n], \quad \sum_S x_{i,S} \leq 1 \tag{3}$$

$$\forall i \in [n], \quad S \subseteq \Omega, \quad x_{i,S} \in \{0, 1\} \tag{4}$$

If we relax integrality constraints of the program we get a linear program whose dual, $P2$, is:

$$\min\left\{\sum_i^n \pi_i + \sum_{j \in \Omega} p_j\right\} \quad (5)$$

$$s.t. \quad \forall i \in [n], S \subseteq \Omega, \quad p(S) + \pi_i \geq v_i(S) \quad (6)$$

$$\forall j \in \Omega, i \in [n], \quad p_j, u_i \geq 0 \quad (7)$$

Bikhchandani and Mamer [6] observed that a Walrasian equilibrium exists iff the value of the maximum social welfare equals the optimum in the problem $P2$. Namely the integrality gap of the LP relaxation of $P1$, is 1. More over, in this case, the set of the optimal dual variables $\{p_j\}_{j \in \Omega}$ is a Walrasian price vector.

By the dual constraints we can bound the dual variables π_i from below by the players' utility $u_{i,p}$. Since we wish to minimize these values and the utility is defined by the prices, we switch the objective to finding a price vector that minimizes the following.

Definition 2.2 (Lyapunov:). $L(p) = \sum_i u_{i,p} + \sum_j p_j$

We usually consider valuations that are rationals. In this case, since the (integer) linear program $P1$ is invariant to scaling of v_i , $i \in [n]$, and $P2$ is invariant to scaling of v_i , $i \in [n]$ and p by the same factor, we may assume that $v_i, i \in [n], p, \pi$ are integers when convenient.

We look at the space of functions on Ω as ordered by the domination order, namely, for $p, q : \Omega \mapsto \mathbb{R}_+$, $p \leq q$ if for every $j \in \Omega$, $p(j) \leq q(j)$. If $p \leq q$ we also say that p is dominated by q .

The *gross substitute* class of valuation is the class in which a player never drops an item whose price was not increased in an ascending auction dynamics. Formally:

Definition 2.3 (gross substitute (*GS*)). For two price vectors p, q and a set S , let $S^=(p, q) = \{j \mid j \in S, p(j) = q(j)\}$. A valuation v is gross substitute if for every price vector p and $S \in D(p)$, for every price vector $q \geq p$, $\exists S' \in D(q)$ such that $S^=(p, q) \subseteq S'$.

In other words, if S is a demand set for p , and $q \geq p$ then there is a demand set for q that contains all elements $i \in S$ for which $p(i) = q(i)$.

It is known [15] that when valuations are monotone the class gross substitute is equivalent to the class of *single improvement* valuations which is defined by the following.

Definition 2.4 (single improvement). A valuation v is in the class single improvement if for every price vector p and a set of items $S \notin D(p)$, there exists a set T , such that $u(S) < u(T)$, $|T \setminus S| \leq 1$ and $|S \setminus T| \leq 1$.

The class of gross substitute valuations is important and has been extensively studied. It can be showed that gross substitute valuations are submodular (see formal definition below). It contains additive valuations (in which every single item has a value, and the value of a set is the sum of values of its items). For some references on gross substitute valuations see [18, 7]. In the context of this study, the importance of gross substitute valuations stems from the following theorem, and the corresponding ascending-auctions of Gul-Stacchetti [15] and Ausubel [2].

Theorem 2.5. [16] *If a set of valuations $\{v_i\}_1^n$ is gross substitute, then it posses a Walrasian equilibrium.*

Moreover, there is an additional nice feature to it. Finding a Walrasian equilibrium is computationally easy if one has the access to the full representation of the valuations, or even a demand oracle of the valuations, by solving the LP relaxation to $P1$ above. However, this full knowledge may be too large to handle, or, and more important, in a real economic situation, not available. It turns out, however, that there is a natural way to compute a Walrasian equilibrium for gross substitute valuations, along with a Walrasian allocation. This is done by a process that is called an ‘‘ascending-auction’’. In an ascending-auction the auctioneer starts from the price vector $p_0 = 0$ and at each step t , finds whether p_t is Walrasian, or increases the prices of some elements to obtain the next price vector p_{t+1} . Thus such a process makes sense also as real economical process.

The notion of ascending-auctions is not well defined. One can potentially start with $p_0 = 0$, compute the Walrasian equilibrium and Walrasian price vector p^* and just set $p_1 = p^*$ on the very next step. A natural ascending-auction should be such that the next step can be decided from the current step with a very limited knowledge of the individual valuations (e.g., at step t only access to $D_i(p_t), i = 1, \dots, n$ should be used), and the increase in prices should be 'natural' namely, can be shown to be required in order to arrive at an envy-free allocation. It turns out that for gross substitute valuations such natural auctions have been proposed [15, 2]. This will be one of the focuses of the study here.

We end this section with the definition of the class $GGGS(k, M)$.

Definition 2.6 (truncation). *For an integer k and $M \in \mathbb{R}_+$, A function (valuation) $v : 2^\Omega \mapsto \mathbb{R}_+$ is a (k, M) -truncation of a valuation u if $v(S) = u(S)$ for every $S \subseteq \Omega$ for which $|S| < k$, and $v(S) = M$ for if $|S| \geq k$.*

Definition 2.7 ($GGGS(k, M)$). *Let k be a natural number and $M \in \mathbb{R}_+$. A valuation $v : 2^\Omega \mapsto \mathbb{R}_+$ is $GGGS(k, M)$, if there is a gross substitute valuation g on Ω such that v is the (k, M) -truncation of g .*

In addition, in order that v will be monotone submodular we further demand that for every S , $g(S) \leq M$ if $|S| < k$ and $g(S) \geq M$ if $|S| \geq k$.

It is quite straightforward that $GGGS(k, M)$ is submodular for every k and M as restricted above. Furthermore, it is obvious that any gross substitute valuation g is in $GGGS(k, M)$ for some k and M - simply, define $k = m + 1$, and $M = \max\{g(S), S \subseteq \Omega\}$. Hence, in this respect, the union of $GGGS(k, M)$ over all k and suitable M 's contains gross substitute. $GGGS(1, M)$ is just the constant valuation function. $GGGS(2, M)$ is already quite interesting. A valuation in this class is arbitrary on singletons, and is M on all pairs. We will show in what follows that $GGGS(2, M)$ is not gross substitute. Moreover, the 'blocking' type obstacle that is used for gross substitute auctions to show that a price vector is not equilibrium (not optimal for the dual LP, $P2$ above), does not hold anymore.

2.1 Basic Structural Results for Gross Substitute

We present here some structural results on gross substitute valuations. These results will turn useful for the simple proof of an ascending auction for gross substitute. Some of these results are new, others might have been known. We include proofs of these properties in the appendix, for completeness.

For a price vector p we denote by $D_i^*(p)$ the collection of minimal demand sets, that is,

$$D_i^*(p) = \{S \mid S \in D_i(p), \forall T \subset S, u_{i,p}(T) < u_{i,p}(S)\}$$

Lemma 2.8 (Matroidity of GS). *If v_i is gross substitute then for each p , $D_i^*(p)$ forms the bases of a Matroid.*

Definition 2.9. *For a set S and a player i , let $f_{i,p}(S) = \min_{D \in D_i^*(p)} \{|D \cap S|\}$.*

For a set $S \subseteq \Omega$, we denote by 1_S the vector which has value 1 for every element $j \in S$ and 0 otherwise. When $S = \{j\}$ is a singleton we use also 1_j .

Lemma 2.10. *Let p be an integer price vector, S a set of items and $p' = p + 1_S$. For a gross substitute player i , $u_{i,p} = u_{i,p'} + f_{i,p}(S)$.*

Definition 2.11. *For two price vectors $p, q \in \mathbb{R}_+^n$, $\max(p, q)$ is the vector that in every coordinate $i \in [n]$, has $\max\{p(i), q(i)\}$. Similarly $\min(p, q)$ is defined.*

Lemma 2.12 (Lyapunov's sub-modularity). [2] *For gross substitute valuations, the Lyapunov is 'submodular' w.r.t. price vectors and the min / max operations. That is,*

$$L(\max(p, q)) + L(\min(p, q)) \leq L(p) + L(q).$$

3 Auctions for Gross Substitute

We review here the ascending auctions of Gul-Stacchetti [15] and Ausubel [2]. The original proofs for the correctness of these auctions were long and complicated. We give here a short proof for this auction. We also show that although the auctions in [15] and [2] are different in description, these two auctions are actually the same.

In this section we assume that all valuations are gross substitute integer functions.

3.1 Witnesses for non existence of an allocation

In the following set of definitions the intention is to isolate a witness showing that a price vector p does not support an envy-free allocation. Recall the definition of $f_{i,p}(S)$ from Definition 2.9.

Definition 3.1. [15] *Let p be a price vector.*

- For a set S , let $f_p(S) = (\sum_{i \in [n]} f_{i,p}(S)) - |S|$.
- Let $O_p^* \subseteq \Omega$ be a set such that $\forall S, f_p(S) \leq f_p(O_p^*)$ and $\forall S \subset O_p^*, f_p(S) < f_p(O_p^*)$. Namely, O_p^* is the set of maximum f_p and that is minimal in this respect.
- If for all sets S , $f_p(S) \leq 0$ we define $O_p^* = \emptyset$.

Again, when the price vector is known we omit the subscript and use $f_i(S)$, $f(S)$ and O^* . The proof of the next observation is in the Appendix.

Observation 3.2. *If $\exists S$ such that $f(S) > 0$ then there is no envy-free allocation.* ■

Hence, in view of Observation 3.2, if no allocation exists due to a set S with $f(S) > 0$, then O_p^* is the extremal obstacle for p to support an envy-free allocation. In particular, the elements in O_p^* are *over demanded* in the sense that in every possible allocation w.r.t p , the number of elements in O_p^* that need to be allocated is more than its size. It thus makes sense to increase the price of these elements, so that they will be less demanded. This is just what the Gul-Stacchetti auction does. In the proof of correctness, one will need to also show that when no obstacle exists, then there is indeed an envy-free allocation, and moreover, that this is in addition a Walrasian allocation (namely, that every positive priced item is allocated).

We state here an additional property (see appendix for proof) of gross substitute valuations.

Lemma 3.3 (Monotonicity). *Let $p' = p + 1_j$, then $\forall S$ for which $j \notin S$, $f_{p'}(S) \geq f_p(S)$*

3.2 Equivalence

Although Gul and Stacchetti's auction is ascending while Ausubel's can ascend or descend, we will show their equivalence. Namely, if we start with the 0 price vector, then at every step the same set of items is chosen for a price increase in both auctions. We next describe both auctions formally.

Gul and Stacchetti's auction:

- Start with $p = 0$
- While $f_p(O_p^*) \neq \emptyset$ increase $p(j)$ by 1 for all $j \in O_p^*$

Ausubel's auction:

- Start with p
- Let $L(\cdot)$ denote the Lyapunov function. While $\exists S, S'$ such that $L(p + 1_S - 1_{S'}) < L(p)$ find S^*, S'^* such that $\forall T, T', L(p + 1_{S^*} - 1_{T'}) \leq L(p + 1_T - 1_{T'})$ and $\forall T \subset S^*, T' \subset S'^*$ it holds that $L(p + 1_{S^*} - 1_{S'^*}) < L(p + 1_T - 1_{T'})$. Increase $p(j)$ by 1 for all $j \in S^*$ and decrease it by 1 for all $j \in S'^*$

We assume in what follows that the auction has a Walrasian equilibrium (as every gross substitute does). Note that for Ausubel's auction, if S, S' as above exist it immediately shows that p cannot be optimal in the dual LP $P2$. If such S, S' are found, Ausubel's auction improves the Lyapunov by the suggested change in price. Hence, the auction is a dual-auction. The actual S^*, S'^* that are chosen are these that maximize the change in the dual cost L .

It was proved by Ausubel that if the current price vector is not higher than equilibrium, this process will not decrease any price. Hence the auction can be described in a simpler way. Define the *minimal minimizer* set $S^*(= S_p^*)$ to be such that $\forall S, L(p + 1_{S^*}) \leq L(p + 1_S)$ and if for a set $S \neq S^*$, $L(p + 1_{S^*}) = L(p + 1_S)$ then $|S^*| \leq |S|$. Clearly, a minimal minimizer always exists, but furthermore, Ausubel provides also a proof of it being unique.

Ausubel's ascending auction:

- Start with $p = 0$
- While $S^* \neq \emptyset$ increase $p(j)$ by 1 for all $j \in S^*$

What we show next (proof in appendix) is that the set S^* in Ausubel's auction collides with O^* in Gul-Stacchetti's.

Corollary 3.4. (of Lemma 2.10) *The two algorithms collide on gross substitute valuation.*

We end this section with the remark that the equivalence shown above is in direct analog to the divisible goods scenario. For auctions in the divisible good scenario a similar Lyapunov is used to identify the set of prices to be changed. The reason is that its sub-gradient at p equals the excess supply, which, in turn, is the analog to the discrete function $f(\cdot)$ (or $-f(\cdot)$ to be precise). See [22] for divisible good combinatorial auctions.

3.3 A finer ascending auction for Gross Substitute

We conclude this section with a 'finer' version of the Gul-Stacchetti auction. The idea is the same, the only difference is that we go by finer steps, namely, at each step we increase the price of a single element. We also include a full proof that it reaches the optimal equilibrium. For this we use the properties that we have stated in Sections 2 and 3.1 with the additional fact that gross substitute valuations possess Walrasian equilibrium (which is not proved here).

Auction:

- Start with $p = 0$
- While $O^* \neq \emptyset$ increase $p(j)$ by 1 for some $j \in O^*$

In order to prove that this algorithm finds an optimal price vector we will show that it never goes above any optimal price vector, and when it stops the dual objective value is at-most the value of the optimal price dual objective. We assume that there is an optimal Walrasian equilibrium.

Let p^* be an optimal price vector, p_t the current price vector and T the last step of the algorithm, that is, we need to show $p^* = p_T$. The next proposition, which its proof is in the appendix, shows exactly that.

Proposition 3.5. *For every $t \leq T$, $p_t \leq p^*$.*

Claim 3.6. *If for an integral price p , $f_p(S) \leq 0$ for some S , then $L(p + 1_S) \geq L(p)$*

Proof. This is an immediate corollary of lemma 2.10. ■

Proposition 3.7 (Optimality). $L(p_T) \leq L(p^*)$.

Proof. By proposition 3.5 $p_T \leq p^*$. Let $O^< = \{j \mid p_T(j) < p^*(j)\}$ be the elements that have smaller price in p_T than in p^* . since T is the last step, $f_T(S) \leq 0$ for every set S , and in particular for $f_T(O^<) \leq 0$. Hence by Claim 3.6 $L(p_T + 1_{O^<}) \geq L(p_T)$.

Now, if $p_T + 1_{O^<} = p^*$ this is in contradiction with the optimality (and minimality) of p^* . Otherwise if $p' = p_T + 1_{O^<} \neq p^*$, the same argument applies to p' . This is true as one can argue that $f_{p'}(S) \leq 0$ for every $S \subseteq O^<$ since this was true for p_T and increasing prices in $O^<$ by 1, cannot cause the increase in $f_{i,p}(S)$ for any set $S \subseteq O^<$. ■

We note that a different proof for the last Lemma is possible along the following lines: By the Matroid union theorem, it can be shown that p_T posses an envy-free allocation, since $O^* = \emptyset$ w.r.t p_T . Proposition 4.13 then shows that this is a Walrasian allocation. However, this uses heavier machinery from Matroid theory.

4 $GG S(k, M)$ valuations

Recall, a valuation v is $GG S(k, M)$ if there is a gross substitute valuation g such that $v = g$ for all sets of size less than k - these will be called *small sets*, and is equal M for larger size sets. $GG S(k, M)$ are submodular, but are not gross substitute in general. Such a small example, showing that $GG S(2, M)$ is already not gross substitute is in Appendix Section A.1. In the following discussion we will show this, by utilizing the facts we know about an auction defined on an ensemble of gross substitute valuations. In particular, this will show that the condition for non existence of an allocation that is used for gross substitute is not useful for $GG S(k, M)$ auctions anymore.

We will consider here combinatorial auctions in which all players have valuations that are in $GG S(k, M)$, namely the same k, M for every player. Note that this does not mean that all valuations are the truncation of the same gross substitute valuation g .

Recall that for gross substitute valuations, the existence of a non-empty set O_p^* was a witness of non-optimality of p . Furthermore, O_p^* was the handle in which ascending auctions like Gul-Stacchetti's and Ausbel's found a set of items whose prices are to be increased. In the next claim we show that this is not the case for some valuations in $GG S(2, M)$. In particular, this shows that $GG S(k, M)$ is not gross substitute already for $k = 2$.

Let n be large enough and $m = 2n - 2$. (e.g., $n = 5, m = 8$ will do). Let $M = 2$ and the following $GG S(2, 2)$ valuations: $v_i, i \leq n - 2$ is uniform value 1 on all singletons. $v_{n-1} = v_n$ assign value 1 for all items but the last which has value 2.

Claim 4.1. *For the valuations above and $p = 0$ there is no allocation, while $\forall S, f_p(S) \leq 0$.* ■

4.1 $GG S(2, M)$ posses Walrasian equilibrium

Here we show that $GG S(2, M)$ always posses Walrasian equilibrium. We don't know whether this fact is true for $GG S(k, M)$ for $k \geq 3$.

Theorem 4.2. *Let Ω be a set of m items and let $v_i, i = 1, \dots, n$ be valuations on Ω , each in $GG S(2, M)$. Then there is a Walrasian equilibrium for v_1, \dots, v_n .*

Before we prove Theorem 4.2, we prove the following intermediate result.

Lemma 4.3. *Let Ω be a set of m items and let $v_i, i = 1, \dots, n$ be valuations on Ω , each in $GG S(2, M)$. Let p be a minimal optimal price vector (namely, p is optimal in the dual LP, P2). Then p poses an envy-free allocation.*

We will recurrently use the following standard fact from matching theory [17].

Fact 4.4.

1. Let $G = (V, E)$ be a graph and assume that $V' \subseteq V$ is matched by some (not necessarily maximum) matching, then there is a maximum matching that matches V' .
2. Let $G = (X, Y : E)$ be a bipartite graph and M a maximum matching. Then (König Egerváry theorem, see e.g., [17]) there is a vertex cover $C \subseteq X \cup Y$ covering the edges of G with $|C| = |M|$. Moreover, M matches the $|C \cap X|$ members of $X \cap C$ with items in $Y \setminus C$ and it matches the $|C \cap Y|$ members of $Y \cap C$ with members of $X \setminus C$.

Of Lemma 4.3. Obviously, to show the existence of an envy-free allocation it is enough to consider only sets in the demand set of each player that are of size at most two (as larger sets don't have larger value). Thus a demand set that does not contain the empty set typically contains some singletons and sets of size 2 (referred here as pairs). Let p be an optimal solution to the associated LP, $P2$ for the valuations v_i , $i = 1, \dots, n$. In the rest of the proof we partition the players into two sets. The set S of players for which the demand set contains *only* singletons, and the set P . Hence the demand set of every player in P contains pairs too (it might also contain singletons). We also partition the items into two sets, the set MIN of items of min price, and the set A of other items. Namely $MIN = \{j \in \Omega \mid \forall j' \in \Omega, p(j) \leq p(j')\}$.

We will recurrently use the following bipartite graph $G = ([n], \Omega : E)$, where $E = \{(i, x) \mid \{x\} \in D_i(p)\}$.

Claim 4.5. *If $|MIN| \geq 2$ then for every player $i \in P$, the demand set $D_i(p)$ contain all pairs in MIN and no other pairs.*

Proof. This is obvious as if the minimum price is p_0 , then the utility of every pair of items from MIN is $M - 2p_0$ while every other pair has smaller utility. ■

Following similar reasoning we get the following claim.

Claim 4.6. *Assume that $MIN = \{x\}$, namely $|MIN| = 1$. Let $MIN2 = \{y \in \Omega \mid \forall z \neq x, p(y) \leq p(z)\}$. Namely, $MIN2$ is the set of items of minimal price in $\Omega \setminus \{x\}$. Then for every $i \in P$ its demand pairs are $MIN \times MIN2$.* ■

Claim 4.7. *There is an allocation of singletons to the players in S .*

Proof. We only need to show that there is a perfect matching in G w.r.t. S .

Indeed, if there is no perfect matching then by Hall marriage theorem there is a set of players $S' \subseteq S$ such that $N(S') = \{x \mid \exists i \in S', (i, x) \in E(G)\}$ has cardinality $|N(S')| < |S'|$. In that case consider an increase of $\epsilon > 0$ in the price of every element in $N(S')$. The total price will increase by $\epsilon \cdot |N(S')|$ while the total utility will decrease by at least $\epsilon \cdot |S'|$. Hence the Lyapunov will decrease in contradiction with the assumption that p is an optimal price vector. ■

Claim 4.8. *If $|MIN| \geq 2$ then there is an allocation of singletons that covers all A and S .*

Proof. We first show that there is a matching M_A in G that matches A . Then, since Claim 4.7 asserts the existence of a matching M_S that matches S , it is easy to see that $M_S \cup M_A$ contains a matching that covers $A \cup S$.

Indeed, assume that there is no matching in G that matches A . In this case, again by Hall theorem, there is a set of items $A' \subseteq A$ for which $|N(A')| < |A'|$, where $N(A') = \{i \mid \exists x \in A', (i, x) \in E\}$. Consider the decrease in price by $\epsilon > 0$ of every element in A' . ϵ is taken small enough so that the elements of minimum price will not be affected. Then the total price will decrease by $\epsilon|A'|$. The utility of players from singleton increases only for players in $N(A')$, and in this case, increases by ϵ . The utility of pairs does not increase at all (by Claim 4.5, as we assume that $|MIN| \geq 2$). Hence in total the Lyapunov decreases in contradiction with the optimality. ■

We now end the proof of the lemma for the case $|MIN| \geq 2$ by the following argument. By Claim 4.8 there is a matching in G that cover S and A . Let M be a maximum matching in G that matches A and S . By Fact 4.4 we may assume the existence of such maximum matching.

If M does not define an envy-free allocation of all players, there is a set of players $P' \subset P \cup S$ for which $|N(P')| < |P'|$. It is standard that we may assume in addition that M leaves unmatched $|P'| - |N(P')| = n'$ players. As M matches S , these unmatched players are in P and in particular, have any pair in MIN in their demand sets.

Assume m' elements in MIN remain unmatched by M . If $2n' \leq m'$ then the allocation defined by M can be augmented by a collection of n' disjoint pairs from the unmatched elements in MIN , and we arrive at an envy-free allocation. Hence we may assume that $2n' > m'$.

Consider the following increased price vector \tilde{p} : for every element in $N(P')$ the price is increased by 2ϵ and for every other element by $\epsilon > 0$ for some small enough ϵ .

We claim that the Lyapunov has decreased in contradiction with the optimality of p . Indeed, $L(\tilde{p}) - L(p) = \sum_{i \in [n]} (u_{i,\tilde{p}} - u_{i,p}) + \sum_{x \in \Omega} (\tilde{p}(x) - p(x))$. To show that this difference is negative, we will show that for each matched player (by M), its difference in utility balances off the difference in the price of the element it is matched to. We then will show that for the other players, the difference in utility (which is negative), is more in absolute value than the difference in the price of unmatched items.

Indeed, for every player in P' , The difference in the price of its item as well as the difference in its utility are 2ϵ . This is true as such player has singletons only in $N(P')$ (and possibly pair, but the utility from a pair drops by at least 2ϵ).

For player not in P' , obviously the difference in utility as well as the difference in the price of its matched item, in absolute value is ϵ .

The total unaccounted difference in price is contributed by the unmatched items. This is, by our notations, $\epsilon \cdot m'$. The total unaccounted difference in utility is contributed by the unmatched players. Note that all unmatched players are in P' . For any such player $x \in P'$ (there are n' such unmatched players), the change in utility is 2ϵ , as singletons have increased price of 2ϵ while pairs have increased price of at least 2ϵ . Hence, in total, by our assumption that $2n' > m'$ we get that the total Lyapunov has decreased.

For the case where $MIN = \{x\}$, Namely $|MIN| = 1$, an analysis similar in nature to this above shows that there is an envy free allocation. The detailed argument for this case is presented in the appendix. ■

of Theorem 4.2. Let p be an optimal solution to the associated LP, $P2$ for the valuations $v_i, i = 1, \dots, n$. Here we assume further that p is the minimal such optimal price vector (w.r.t. the order on prices as defined in the preliminaries).

We show that p poses a Walrasian allocation rather just an envy-free one. We consider the same graph G as in the proof of Lemma 4.3. Assume first that $|MIN| \geq 2$. In this case Claim 4.8 asserts that there is a matching in G that matches $S \cup A$. Hence, by Fact 4.4 there is a maximum matching M in G that matches $S \cup A$. We assume that this matching leaves n' players and m' items unmatched. Moreover, Lemma 4.3 asserts that there is a envy-free allocation, hence $2n' \leq m'$ as the unmatched players are allocated disjoint pairs from MIN . We denote this set of m' unmatched items by M' and the set of n' unmatched players by N' .

To understand the situation, disregarding M , we first consider a maximum matching M_{SA} between S and A . By Fact 4.4 we know that there is a cover $C_S \cup C_A$ of all edges going from S to A , where $C_S \subseteq S$, $C_A \subseteq A$ and $|C_S| + |C_A| = |M_{SA}|$. In addition, it follows that M_{SA} leaves $r = |A| - (|C_S| + |C_A|)$ items unmatched in A , and $d = |S| - (|C_S| + |C_A|)$ players from S . Note also that since $C_S \cup C_A$ is a cover, it follows that there is no edge between $S \setminus (C_S)$ and $A \setminus C_A$.

We now turn our attention back to the matching M . M matches $S \cup A$ (and some other items). It need not be consistent with the matching M_{SA} , but as it matches all players in S and all items in A , it must use extra r players from P and extra d items from MIN .

Now, consider the price vector \tilde{p} that is obtained from p as follows: We reduce the price of every item in $A \setminus C_A$ by 2ϵ and reduce the price of any other item by ϵ . Hence the total decrease in price $-\Delta(\tilde{p}) = p(\Omega) - \tilde{p}(\Omega) = \epsilon(m' + d + |C_A| + 2|C_S| + 2r)$.

To analyze the change in utility, note that every player in P has an increase in its utility by 2ϵ . This is also the case for players in C_S . However, note a player in $S \setminus C_S$ has an increase of ϵ , as its demand contains only singletons from $C_A \cup MIN$. Hence the total increase in utility is $\Delta(\tilde{u}) = \sum_i u_{i,\tilde{p}} - u_{i,p} \leq 2\epsilon n' + 2\epsilon r + 2\epsilon|C_S| + \epsilon|C_A| + \epsilon d$.

Comparing the decrease in price and increase in utility, taking into account that $2n' \leq m'$ we conclude that $\Delta(\tilde{u}) \leq -\Delta(\tilde{p})$ which implies that $L(\tilde{p}) \leq L(p)$ which is a contradiction.

In the case $|MIN| = 1$, Claim 4.6 implies that all pairs in the demand sets intersect (in MIN). Since there is an envy free allocation by Lemma 4.3, it follows that at most one player can be allocated a pair. We conclude that either $n - 1$ items from A are allocated as singletons, plus one pair, or n items are allocated as singletons. In any case, if the allocation is not Walrasian it follows that $|A| \geq n$ and hence decreasing prices in all items in A by ϵ will not increase Lyapunov. ■

4.2 An Ascending auction for $GG(2, k)$

We propose here an ascending-auction for $GG(2, k)$. This auction is a natural generalization of induced gross substitute auction on unit-demand valuations of which the $GG(2, k)$ valuations are the $(2, M)$ -truncation. We will prove that this auction terminates in a Walrasian allocation, and a corresponding Walrasian equilibrium.

Recall that for $GG(2, k)$ valuations, and a price vector p , we may restrict the demand sets of each player to its demanded sets of size at most two. Again, for a price p we will partition the players into those that have only singletons in their demand sets. These players will be called 'small' players. The other players have pairs in their demand sets (and may also have singletons). We disregard players whose demand sets contains the empty set.

We note that given a price vector p , we may consider the auction induced only on the small players S , and corresponding singletons $N(S)$. The demand sets for this induced auction are consistent with the unit-demand valuation (which is gross substitute), and will be referred as the induced gross substitute auction on S . We considered several gross substitute ascending auctions, in what follows we refer to the auction of Gul-Stacchetti in which a price of all items in the over-demanded set O^* is increased at every step.

We assume in what follows, as discussed before, that all valuations are integral, and that the minimum Walrasian equilibrium price vector is also integral.

Auction:

1. Start with $p = 0$
2. If the set of small players $S \neq \emptyset$, and there is no allocation for this set, increase prices by taking one step of the induced gross substitute auction on these players. Note that as a result some small players may cease to be small. Go back to 2.
3. There is a perfect matching (partial allocation) of the small players to items. Find a maximum matching of players to singletons that matches all small players. Assume that n' players and m' items remained unmatched.
4.
 - If $2n' \leq m'$ then p is a Walrasian price vector.
 - Else ($2n' > m'$), increase the price of all items in MIN by 1 and go back to 2. with the new price vector and the empty allocation.

Theorem 4.9. *Let v_1, \dots, v_n be $GG(2, M)$ valuations. Then the auction above stops at a Walrasian allocation, and a Walrasian equilibrium.*

Proof. It is obvious, by the definition of the auction that the price vector is integral at every step. Theorem 4.2 asserts that there is a Walrasian equilibrium. Let p^* be an integral Walrasian price vector. Propositions 4.10, 4.12 and 4.11 imply that the proposed auction never increases the prices beyond p^* , and that there exists an envy free allocation when it stops. Proposition 4.13 implies then that the allocation is Walrasian. ■

Proposition 4.10. *Let p be a current price vector in the auction above, for which $p \leq p^*$. If there is no allocation of the small players w.r.t. p , and hence an increase in price is done in step 2., then the resulting price vector $p + 1_{O^*} \leq p^*$.*

Proof. Recall that the prices in the gross substitute auction that is induced on the small players increases the price by 1 for every item in O_p^* where O_p^* is the minimal most over-demanded set. Now, assume for the contrary that $p + 1_{O^*}$ is not dominated by p^* . Then (using the integrality assumption) there must be a non empty set $O^= \subseteq O_p^*$ of items for which $\forall i \in O^=, p(i) = p^*(i)$. Moreover, $O^* \neq O^=$ as otherwise, $f_{p^*}(O^*) \geq f_p(O^*) > 0$. This is so since if $O^* = O^=$ then $\forall i \in O^*, p^*(i) = p(i)$, while all items outside O^* have non smaller prices in p^* . Hence the demand sets of all the players that contribute 1 to $f_p(O^*)$ remain unchanged w.r.t. p^* .

Let B the set of players that have demand sets in $O^=$ and that have no demand set outside O^* . We first claim that $|B| > |O^=|$. Indeed, otherwise, $f_p(O^* \setminus O^=) \geq f_p(O^*)$ which is in contradiction to either the minimality or extremity of O^* .

We conclude that w.r.t. p^* , the set of players B have their demand sets only in $O^=$ (as the prices of these items did not increase w.r.t. p while the prices of other items in O^* did increase). Hence $O^=$ is over-demanded set w.r.t. p^* in contradiction that p^* is Walrasian. ■

Proposition 4.11. *Let $p \leq p^*$ be a current price vector, assume further that there is an allocation for the small items, and that price is increased in step 4. due to the fact that $2n' > m'$, then the resulting price is still dominated by p^* .*

Proof. Assume that $p \leq p^*$ and that contrary to the proposition prices are to be increased in step 4. resulting in the price $p' = p + 1_{MIN}$ which is not dominated by p^* any more. Hence there is a non empty set $O^= \subseteq MIN$ of items, for which $\forall i \in O^=, p(i) = p^*(i)$.

We consider several cases here. Assume first that $|O^= \cap MIN| \geq 2$. In this case, all players that had pairs in their demand set w.r.t. p will still have all pairs in $O^= \cap MIN$ in their demand set w.r.t. p^* and with the same utility. For small players, since in p^* they are allocated, their change in utility balances off the change in price of the items they are matched with. Thus, altogether, since utility matches off change in price for small players, utility has remained the same for non small players, while total price has increased, it must be the $L(p^*) \geq L(p)$ which is in contradiction to the optimality of p^* .

Consider now the case in which $O^= \cap MIN = \{x\}$. Let $MIN2 = \{y \in \Omega \mid \forall z \neq x, p^*(y) \leq p^*(z)\}$. Namely, $MIN2$ is the set of items of minimal price w.r.t. p^* in $\Omega \setminus \{x\}$. In this case, w.r.t. p^* , all players that are not small want all pairs in $\{x\} \times MIN2$. Hence, only one such player can be allocated a pair, and if there is such player, its utility increased by $\alpha = p^*(y) - p^*(x)$ w.r.t. its utility in p .

Recall also that in p^* there is a Walrasian allocation. Hence, this allocation assigns singletons to all players possibly except one as explained above. Consider now the price vector p^{**} obtained from p^* by decreasing the price of every item except x , by α . By the discussion above, the allocation w.r.t. p^* is still a valid allocation w.r.t. p^{**} . Hence p^{**} is a Walrasian equilibrium in contradiction with the minimality of p^* . ■

Proposition 4.12. *If $2n' \leq m'$ then p is a Walrasian price vector.*

Proof. In this situation there is an envy-free allocation (as the n' players can be assigned arbitrary disjoint pairs from the remaining items). Since by induction $p \leq p^*$, Proposition 4.13 implies that p is Walrasian. ■

Proposition 4.13. *Let p^* be Walrasian equilibrium for an arbitrary combinatorial auction (not necessarily submodular). Let $p \leq p^*$ so that there is an envy-free allocation with respect to p . Then p is Walrasian equilibrium.*

Proof. Let (S_1^*, \dots, S_n^*) be the Walrasian allocation w.r.t. p^* and let (S_1, \dots, S_n) be the envy-free allocation w.r.t. p . Let $S = \Omega \setminus \cup_{i=1}^n S_i$ be the set of unallocated items. We get,

$$\begin{aligned} L(p) &= p(S) + \sum_i (u_{i,p}(S_i) + p(S_i)) = \\ &\sum_i (u_{i,p^*}(S_i) + p^*(S_i)) - (p^*(S) - p(S)) \leq \\ &\sum_i u_{i,p^*}(S_i^*) + \sum_i p^*(S_i^*) = L(p^*) \end{aligned}$$

Where the second equality is since the drop in utility of the set S_i is just the increase in the prices of the corresponding items. The first inequality is since S_i is not necessarily in the demand set of i w.r.t. p^* , and since $(p^*(S) - p(S)) \geq 0$.

Hence, since $L(p) \leq L(p^*)$ equality must occur and p is Walrasian too. ■

5 Further Discussion - Witnesses for the non-existence of an allocation

As already discussed before, a crucial feature of gross substitute auctions, that allowed an ascending auction is the existence of a condition that characterize non envy-free prices. Namely, as shown, a price p does not posses envy-free allocation if and only if there is a set of over demanded items, namely $O_p^* \neq \emptyset$. One reason is that assuming that a current non envy-free price p satisfies $p \leq p^*$ for a Walrasian (or even just envy-free) p^* directs the auction to what prices should be increased.

For valuation in $GGs(2, M)$, the condition above was shown not to hold, but yet, another characterization was developed (although not explicitly stated). Essentially, this can be read off the proof of Lemma 4.3. It states that p is envy-free if and only if there is a maximum matching of singletons to players that covers all players in S , and leaves n' unmatched players, m' unmatched items from MIN with $2n' \leq m'$.

Hence, a good starting point for the existence of ascending auction that finds a Walrasian equilibrium if such exists should be the existence of such characterization. This is also interesting from the complexity (namely, the computational resources to decide on next step in an auction) point of view. It puts the decision whether a given price vector posses envy-free allocation in $NP \cap CO - NP$ (here the input is taken as the minimal demand sets $D_i^*(p)$, $i \in [n]$).

Such a characterization is not expected for any set of valuations. In fact it can be shown that for some class of valuations, where the demand sets are restricted to size two (and moreover, each demand set contains at most two pairs), to decide whether the 0-price allows envy-free allocation is NP-hard. The same result holds for the well known class of fractionally sub-additive valuations (aka XOS), see [12, 11] for definitions. However, these classes of allocation are not submodular, and we do not know if for any submodular class, the above characterization question is tractable.

For $GGs(k, M)$ an inefficient characterization exists in a sense, (in particular not known to be in $co - NP$ even for constant k). Moreover, we don't know how to use it to construct an ascending auction that converges to a Walrasian equilibrium if exist. This in turn, raises the following more finer decision problem: For gross substitute valuations, as well as $GGs(2, M)$, for a given p that is dominated by some unknown Walrasian price p^* , there was a way to identify an item $x \in \Omega$ for which $p + 1_x$ is still dominated by p^* . We don't know if this can be done for $GGs(k, M)$. If one could do this, we would immediately get the desired ascending auction for $GGs(k, M)$.

6 Conclusions

We present the family of submodular valuation classes $GGS(k, M)$ that generalize gross substitute. We prove that for some non-trivial subclasses of it ($GGS(2, M)$) Walrasian equilibrium always exists and there is a natural ascending auction that reaches this Walrasian equilibrium. This is contrary to the common belief that only gross-substitute valuations have these property.

While open questions w.r.t. $GGS(k, M)$ valuations exist and are implicitly obvious from the discussion above, we reiterate the most interesting open problem in the context of submodular valuations. This is to give a characterization, (or at least to find other significant families) of valuations inside submodular, but that are not gross-substitute and that do have Walrasian equilibrium. An additional question in this generality (while less well-defined) is to present such families that admit *natural* ascending-auctions.

References

- [1] T. Andersson, C. Andersson, and A.J.J. Talman. Sets in excess demand in simple ascending auctions with unit-demand bidders. *Annals of Operations Research*, pages 1–10, 2013.
- [2] L. M. Ausubel. Walrasian tatonnement for discrete goods. Manuscript, 2005.
- [3] L.M. Ausubel. An efficient ascending-bid auction for multiple objects. *The American Economic Review*, 94(5):1452–1475, 2004.
- [4] L.M. Ausubel and P. Milgrom. Ascending auctions with package bidding. *Frontiers of Theoretical Economics*, 1(1):1–42, 2002.
- [5] S. Bikhchandani, S. De Vries, J. Schummer, and R.V. Vohra. An ascending vickrey auction for selling bases of a matroid. *Operations research*, 59(2):400–413, 2011. Preliminary version in SODA’08.
- [6] S. Bikhchandani and J. W. Mamer. Competitive equilibrium in an exchange economy with indivisibilities. *Journal of Economic Theory*, 74(2):385 – 413, 1997.
- [7] E. K. Browning. *Microeconomic Theory & Applications (6th Edition)*. New York: Addison Wesley, 1999.
- [8] R. Colini-Baldeschi, M. Henzinger, S. Leonardi, and M.n Starnberger. On multiple keyword sponsored search auctions with budgets. In *ICALP (2)*, pages 1–12, 2012.
- [9] G. Demange, D. Gale, and M. Sotomayor. Multi-item auctions. *The Journal of Political Economy*, pages 863–872, 1986.
- [10] S. Dobzinski, R. Lavi, and N. Nisan. Multi-unit auctions with budget limits. *Games and Economic Behavior*, 74(2):486–503, 2012. Preliminary version in FOCS’08.
- [11] S. Dobzinski, N. Nisan, and M. Schapira. Approximation algorithms for combinatorial auctions with complement-free bidders. In *STOC*, pages 610–618, 2005.
- [12] U. Feige. On maximizing welfare when utility functions are subadditive. In *STOC*, pages 41–50, 2006.
- [13] G. Goel, V. Mirrokni, and R. Paes Leme. Polyhedral clinching auctions and the adwords polytope. In *Proceedings of the 44th symposium on Theory of Computing (STOC)*, pages 107–122, 2012.
- [14] F. Gul and E. Stacchetti. Walrasian equilibrium without complementarities. *Journal of Economic Theory*, 83:95–124, 1999.
- [15] F. Gul and E. Stacchetti. The english auction with differentiated commodities. *Journal of Economic Theory*, 92(1):66 – 95, 2000.

- [16] A. S. Kelso and V. P. Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, 1982.
- [17] M.D. Plummer L. Lovász. *Matching Theory*. North Holland, 1986.
- [18] W. Nicholson. *Microeconomic Theory: Basic Principles and Extensions*. The Dryden Press, 1998.
- [19] D.C. Parkes. iBundle: An efficient ascending price bundle auction. In *the Proceedings of the first ACM Conference on Electronic Commerce*, pages 157 – 166, 1999.
- [20] N. Sun and Z. Yang. Equilibria and indivisibilities: gross substitutes and complements. *Econometrica*, 74(5):1385–1402, 2006.
- [21] N. Sun and Z. Yang. A double-track adjustment process for discrete markets with substitutes and complements. *Econometrica*, 77(3):933–952, 2009.
- [22] H. R. Varian. Chapter 3 dynamical systems with applications to economics. volume 1 of *Handbook of Mathematical Economics*, pages 93 – 110. Elsevier, 1981.
- [23] L. Walras. *Éléments d'Économie Politique Pure, ou Théorie de la Richesse Sociale*. Corbaz, Lausanne, 1874.

A Proofs for claims in Sections 2 and 3.

In this section we assume that all valuations and prices are integral. This is w.l.o.g. as taking any set of valuations and price vector, scaling all by the same constant does not change the demand sets. As we assume that the valuations are rationals, and by scaling integral, scaling large enough means that prices too can be rounded up to the closest integral value. More over, as remarked in Section 2 the optimum for the dual LP, $P2$ is also integral.

Proof of Lemma 2.8

Proof. Let v be a gross substitute valuation, p a price vector and let $D_1, D_2 \in D_v^*(p)$ be two minimal demand sets. Let p' be the price vector defined by $p'(j) = p(j) \forall j \in D_1 \cup D_2$ and $p'(j) = \infty$ else. Note that D_1, D_2 are minimal demand sets on p' as well. Let $j_2 \in D_2 \setminus D_1$ and define $p'_2 = p' + 1_{j_2}$. Clearly $D_2 \notin D(p'_2)$, since the utility of D_1 is bigger for this price vector. Note also that $u_{p'_2}(D_2) = u_p(D_2) - 1$. By the single improvement property there exists D_3 such that $|D_3 \setminus D_2| \leq 1$, $|D_2 \setminus D_3| \leq 1$ and the utility of D_3 on p'_2 is grater than D_2 . Hence $u_{p'_2}(D_3) > u_{p'_2}(D_2) = u_p(D_2) - 1$ which implies (by the integrality assumption) that $u_p(D_2) \leq u_{p'_2}(D_3) \leq u_p(D_3)$. In particular it follows that $j_2 \notin D_3$, $D_3 \in D_p$ and (by minimality of D_2 w.r.t. inclusion) $D_3 \setminus D_2 \neq \emptyset$.

Hence $\exists j_1 \in D_1 \setminus D_2$ such that $j_1 \in D_3$. We conclude that $D_3 = D_2 \cup \{j_1\} \setminus \{j_2\}$, for $j_1 \in D_1 \setminus D_2$. Note that this conclusion is exactly the replacement property (aka the exchange principle) of a Matroid. By induction on $|D_2 \setminus D_1|$ this also implies that $|D_1| = |D_2|$.

Formally, the only thing left to prove is that the set $D^*(p)$ is not empty. Indeed, for each p , the demand set is well defined and hence the minimal demand set is not empty. ■

Proof of Lemma 2.10

Proof. Clearly $u_p \leq u_{p'} + f_p(S)$ since a set D for which $|S \cap D| = f_p = f$, guaranties a utility $u_{p'}(D) = u_p(D) - f$. Assume that $u_p < u_{p'} + f$, that is, there exists a set S' that gives a better utility in p' . The next claim, implicitly suggests that there exists a base B in p , such that $u_{p'}(B) \geq u_{p'}(S')$. Since this holds for every S' , and $u_{p'}(B) \leq u_p - f$ the lemma follows. ■

Claim A.1 (Utility Distance). *For a gross substitute valuation, let p be a price vector and $u_p(S) + l = u_p$, that is, the utility of S is smaller than optimum utility by exactly l . Then, there are two sets R, D such that $D \subseteq S \cup R$, $|R| \leq l$, $D \in D(p)$.*

Proof. The proof is by induction on l . For $l = 0$ the claim holds by $D = S$ and $R = \emptyset$. Assume that $l \geq 1$ and $u(S) + l = u_p$. By the single improvement property there is a set T and an element j , such that $u(T) > u(S)$ and $T \setminus S \subseteq \{j\}$. By the induction hypothesis, there are two sets R', D such that $D \subseteq T \cup R'$, $|R'| \leq u_p - u(T)$, and $D \in D(p)$. If we now take $R = R' \cup \{j\}$ (or $R' = R$ if $T \subseteq S$) we will get the two sets R, D as needed, as $D \subseteq S \cup R$, and $|R| \leq u_p - u(T) + 1 \leq u_p - u(S) \leq l$. ■

Lemma 2.8 provides information on the demand set at a price vector p . It does not, however, characterize the gross substitute valuations v . In particular, it does not specify how the demand set $D_i(p)$ changes when p changes. The next lemma go in this later direction.

Lemma A.2. *Let p be an integer price vector and let $p' = p + 1_j$ for some element j , then for a gross substitute valuation v , $D^*(p')$ determined by $D^*(p)$ as follows:*

- *If j is not in all members of $D^*(p)$, then $D^*(p')$ contains the members of $D^*(p)$ that do not contain j .*

- If j is in all members of $D^*(p)$ then there might be two cases: In the first, $D^*(p') = \{S \setminus \{j\} \mid S \in D^*(p)\}$, namely, j is deleted from any member of $D^*(p)$, to result in $D^*(p')$.

In the other case, $D^*(p') = D^*(p) \cup D'$ where D' is a collection of some new sets, each of the form $B \cup \{j'\} \setminus \{j\}$, for an old $B \in D^*(p)$ and an element $j' \neq j$.

We note that in Matroid notations, these possibilities are as follows. In the first case, the Matroid $M(p')$ is result of the contraction of j from the Matroid $M(p)$.

For the 2nd case: In the first possibly, $M(p')$ is the result of deleting j from $M(p)$. The 2nd possibly is more complicated, in this case the rank of the Matroid $M(p)$ does not drop, and some new bases are added.

Proof. Clearly if for some $B \in D^*(p)$, $j \notin B$ then $B \in D^*(p')$. Assume j is in every base of $D^*(p)$. Note that it follows that every set in $D(p)$ also contains j . The utility of a base went down by exactly 1 with the increase of j 's price. In particular, by the integrality assumption, every base in $D^*(p)$ is still in $D(p + 1_j)$. Let A be a new base. Since j is in every demand set for p we know that $j \notin A$ and $|A| \leq |B|$ for all bases B . More over, it follows that $u_p(A) = u_p(B) - 1$.

Since $A \notin D(p)$, we apply the single improvement property to obtain a set C with a strictly grater utility and with at most one element more than A . Since $u_p(A) = u_p(B) - 1$ it follows that $u_p(C) = u_p(B)$, namely $C \in D(p)$ and $C \setminus A = \{j\}$. Now either $|A| < |B|$ and hence we conclude that $A = B \setminus \{j\}$ for some base B . Or $|A| = |B|$ and hence $A = B \cup \{j'\} \setminus \{j\}$ for some base B and for every $B \in D(p)$, $B \in D(p + 1_j)$. ■

We now prove Lemma 2.12. As a first step we claim that the following definition is an equivalent definition for functions 'sub-modularity' in the context above. We note that this in complete analogy with the standard relation between submodular functions on the Boolean cube and the standard decreasing marginal property.

Definition A.3. An integer function α on p has the 'decreasing marginal return' property if for every p and $x, y \in \Omega$,

$$\alpha(p + 1_x) + \alpha(p + 1_y) \geq \alpha(p) + \alpha(p + 1_x + 1_y)$$

.

Claim A.4. An integer function α is submodular if and only if α has the decreasing marginal return property.

Of the claim. As usual, note that having the decreasing marginal property is a sub case of sub-modularity. Here we prove the 'other' direction.

Assume that α has the decreasing marginal return property. We prove that α is submodular on every integer p, q by induction on $|p - q|_1$. For $|p - q|_1 = 2$, this is just the decreasing marginal return. Assume then that $|p - q|_1 \geq 3$, and assume w.l.o.g that $p(x) > q(x)$. Then, by the induction assumption,

$$\alpha(\max(p - 1_x, q)) + \alpha(\min(p, q)) \leq \alpha(p - 1_x) + \alpha(q) \quad (8)$$

This is so as $|(p - 1_x) - q|_1 < |p - q|_1$, and $\min(p, q) = \min(p - 1_x, q)$. Equivalently, we get

$$\begin{aligned} \alpha(\max(p, q)) - (\alpha(\max(p, q)) - \alpha(\max(p - 1_x, q))) + \alpha(\min(p, q)) \\ \leq \alpha(p - 1_x) + \alpha(q) \end{aligned}$$

However, by the decreasing marginal return, the parenthesis in the left hand side is less than $\alpha(p) - \alpha(p - 1_x)$, (by taking $s = \max(p, q) > p$). Plugging this to the left hand side implies that

$$\alpha(\max(p, q)) + \alpha(\min(p, q)) \leq \alpha(p) + \alpha(q)$$

■

Of Lemma 2.12: We prove that $L(\cdot)$ has the decreasing marginal return property. Putting the definition in equivalent form, we need to prove that for every p , $L(p + 1_x) - L(p) \geq L(p + 1_y + 1_x) - L(p + 1_y)$.

Note first that $L(p + 1_x) - L(p) \leq 1$ for every price function p . Namely, by increasing a price of an item by 1, the Lyapunov can increase by at most 1, since utilities may just decrease.

Thus, by the observation above, if $L(p + 1_x) - L(p) = 1$ there is nothing to prove. We may assume then (by the integrality assumption) that $L(p + 1_x) - L(p) = -r$ for $r \geq 0$. Consider now Lemma A.2. For a player v_i that has in $D_i^*(p)$ both a base containing x and one that does not contain x , u_i , the utility of i does not change by increasing the price of x . Thus if $r \geq 0$, it must be that at least for one player x is in every base. Moreover, by Lemma A.2, for each such player, the utility decreases by exactly 1. Hence there must be exactly $r + 1$ players for which every base contains x w.r.t. p . Again, by Lemma A.2, for each such player, moving from p to $p + 1_y$ does not change this fact (as a result, some new players may have the same situation, namely that x is in every base w.r.t. $p + 1_y$, but certainly the old ones remain). Hence, moving from $p + 1_y$ to $p + 1_y + 1_x$, the utility goes down by at least $r + 1$, and the Lyapunov goes down by at least r ending the proof. ■

Proof of Observation 3.2

Proof. In every allocation, the i th player gets at least $f_{i,p}(S)$ elements from S , since every demand set of it intersects S by at least this amount. Hence, as an allocation is an assignment of disjoint sets, the total number of elements allocated from S is $s = \sum_i f_{i,p}(S)$. Thus if $f(S) > 0$, namely $s > |S|$ there is no allocation. ■

Proof of Lemma 3.3

Proof. All players that have a base not containing j have less sets to select a minimum from. For players for which the size of members of $D^*(p + 1_j)$ went down w.r.t. to $D^*(p)$, f did not change since $j \notin S$. For players which the size of members in $D^*(p + 1_j)$ did not change, (i.e., have new bases where $D^*(\cdot)$ is viewed as bases of a Matroid), since these new bases are super-sets of the old bases without the element j , the f 's value did not went down too. ■

We prove the equivalence (corollary 3.4) now

Proof. For an auction on gross substitute valuations, that starts with $p_0 = 0$, let O_p^*, S_p^* be the corresponding sets in Gul-Stacchetti's and Ausubel's auctions. All we need to prove is that at every step $O_p^* = S_p^*$.

Let p_w be a minimal price supporting a Walrasian equilibrium. Assume that for a price vector $p \leq p_w$, $O_p^* \neq S_p^*$, then since by the definition of Ausubel auction, S^* is the set that minimize $L(p + 1_{S^*})$, either $L(p + 1_{S^*}) < L(p + 1_{O^*})$, or equality holds. Denote $p_s = p + 1_{S^*}$, and $p_o = p + 1_{O^*}$. In the first case, by the Lyapunov definition, $p(\Omega) + |S^*| + \sum_i u_{i,p_s} < p(\Omega) + |O^*| + \sum_i u_{i,p_o}$. By Lemma 2.10 we have that $|S^*| - \sum_i f_{i,p}(S^*) < |O^*| - \sum_i f_{i,p}(O^*)$. This implies that $f_p(O^*) < f_p(S^*)$ in a contradiction to the maximality of $f(O^*)$. If, on the other hand, $L(p + 1_{S^*}) = L(p + 1_{O^*})$ we will get by the same reasoning a contradiction to the minimality of $|O_p^*|$, since if $f(O^*) = f(S^*)$, by the 'minimal minimizer' definition and uniqueness of S^* , it would imply that $|S^*| < |O^*|$. ■

Proof that the price never goes too high (proposition 3.5)

Proof. Assume that the statement is false and let t be the last time for which $p = p_t \leq p^*$. Let O_t^* be the corresponding obstacle. If the price of all elements in O_t^* already reached their maximum price, then by Lemma [3.3], $f_p(O_t^*) \leq f_{p^*}(O_t^*)$ a contradiction to the fact that there is an envy-free allocation w.r.t. p^* (recall Observation 3.2).

Hence we may assume that there exists some elements in O_t^* for which p assigns a smaller price than p^* . Denote the set of all these elements as O^\neq . Denote the set of the other elements O^\neq , that is, O_t^* is a disjoint union of the sets O^\neq and O^\neq .

Let p' be $p + 1_{O^*}$, then $\min(p', p^*) = p + 1_{O^*}$ and $\max(p', p^*) = p^* + 1_{O^*}$. Lemma 2.12 (sub-modularity) implies that $L(p^* + 1_{O^*}) + L(p + 1_{O^*}) \leq L(p') + L(p^*)$. By the optimality of p^* , $L(p^*) \leq L(p^* + 1_{O^*})$ which implies that $L(p + 1_{O^*}) \leq L(p')$. Expanding the definition of Lyapunov in the last inequality we get, $p(\Omega) + |O^*| + \sum_i u_{i,p+1_{O^*}} \leq p(\Omega) + |O^*| + \sum_i u_{i,p+1_{O^*}}$.

Now by the Lemma 2.10 we conclude that $|O^*| - \sum_i f_{i,p}(O^*) \leq |O^*| - \sum_i f_{i,p}(O^*)$, hence $f_p(O^*) \leq f_p(O^*)$ which is a contradiction either to the minimality of O^* or to the maximality of $f_p(O^*)$. ■

A.1 $GG(2, M)$ is not gross substitute

For a concrete small example showing that $GG(2, M)$ is not contained in gross substitute, consider the following valuation which is in $GG(2, 4)$. There are three items a, b, c where $v(a) = v(b) = 2$, $v(c) = 4$. Consider now the price vectors $p(a, b, c) = (0, 1, 2)$ and $q(a, b, c) = (2, 1, 2)$. The set $\{a, b\} \in D(p)$ since its utility is 3 which is maximum, but in $D(q)$ there is no set containing the element b contrary to the definition of gross substitute valuations (see Definition 2.3).

A.2 Last case for the proof of Lemma 4.3

We present here the proof of the Lemma for the case where $MIN = \{x\}$, Namely $|MIN| = 1$. In this case, let M be a maximum matching in the graph G . We may assume that M is not perfect w.r.t. $P \cup S$ as otherwise, it would define an envy free allocation.

By Fact 4.4 there is a cover of the edges of G $C_1 \cup C_2$, where $C_1 \subseteq (S \cup P)$, $C_2 \subseteq (A \cup \{x\})$, and $|C_1| + |C_2| = |M|$. Also, since the players in C_1 are matched into $(A \cup \{x\}) \setminus C_2$, it follows that

$$|C_1| \leq |(A \cup \{x\}) \setminus C_2| \quad (9)$$

Moreover, since we assume that $|M| < |S \cup P|$, it follows that

$$|C_2| < |(S \cup P) \setminus C_1| \quad (10)$$

Assume first that $x \in C_2$. In that case, increasing prices of items in C_2 by small enough ϵ increases the total price by $\epsilon|C_2|$ while the utility of every player in $(S \cup P) \setminus C_1$ decreases by ϵ (we use here the fact that utility of pairs decreases by at least ϵ as $x \in C_2$). By Equation (10) we conclude that the Lyapunov has decreased.

Consider now the case where $x \notin C_2$. In this case either $m \leq n$ (where $m = |\Omega|$), or $m > n$. In the first case the increase of the price in every item in A by ϵ obviously decreases the Lyapunov (since the utility from of every player decreases by exactly ϵ). In the later case ($m > n$), decreasing the price of every item in A by ϵ decreases total price by $\epsilon(m - 1)$, while every player's utility increases by exactly ϵ . Hence the total Lyapunov does not increase.